ON QUANTIZATION OF TIME-DEPENDENT SYSTEMS WITH CONSTRAINTS

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The Dirac method of canonical quantization of theories with second class constraints has to be modified if the constraints depend on time explicitly. A solution of the problem was given by Gitman and Tyutin. In the present work we propose an independent way to derive the rules of quantization for these systems, starting from physical equivalent theory with trivial non-stationarity.

I. INTRODUCTION

In relativistic particle theories and string theories explicit time-dependent gauges are often used [1]. Not only this problem but also others are bringing out the necessity to formulate in general rules of quantization of time-dependent systems with constraints. The canonical quantization of time-dependent systems with constraints has been formulated by Dirac [2] and described in [3].The generalization of the Dirac method of canonical quantization for the case of time-dependent constraints was described in the book of Gitman and Tyutin [3]. A development of the method and examples can be found in [4]. In this paper the interpretation of two general moments on which Gitman-Tyutin quantization (GT-quantization) is based is given. These are: formal introduction of a momentum ϵ conjugated to the time t, and postulation of a special non-unitary Schrödinger time dependent of operators.

II. GT- QUANTIZATION OF THEORIES WITH TIME-DEPENDENT SECOND-CLASS CONSTRAINTS

Here, we briefly describe the modification of the Dirac method of quantization for time-dependent second-class constraints proposed in [3].

Let us have a theory in a Hamiltonian formulation with second-class constraints $\Phi(\eta,t)=0$, $\eta=(q,p)$, which can explicitly depend on the time t. Then the equation of motion of such a system may be written in the usual form, if one formally introduces a momentum ϵ conjugated to the time t, and defines the Poisson bracket in the extended space of canonical variables $(q,p,t,\epsilon)=(\eta,t,\epsilon)$,

$$\dot{\eta}=\left.\{\eta,\, H+\epsilon\}\right|_{D(\Phi)}, \quad \Phi(\eta,t)=0, \tag{1}$$

where H is a Hamiltonian of the system, and $\{A,B\}_{D(\Phi)}$ is the notation for the Dirac bracket with respect to a system of second-class constraints Φ. The Poisson bracket, wherever encountered, is henceforth understood to be one in such an above mentioned extended space. The total derivative of an arbitrary function $A(\eta,t)$, with allowance made for the equations (1), has the form

$$\frac{dA}{dt}=\left.\{A,\, H+\epsilon\}\right|_{D(\Phi)}.$$

In this case the quantization procedure in the Schrödinger picture can be formulated as follows. The variables η of the theory are assigned the operators $\hat{\eta}$, which satisfy the equal-time commutation relations ($[,\}$ denotes the generalized commutator, commutator or anti-commutator depending on the parities of the variables),

$$[\hat{\eta},\hat{\eta}'\}=\left.i\{\eta,\eta'\}_{D(\Phi)}\right|_{\eta=\hat{\eta}}, \tag{2}$$

the constraints equation

$$\Phi(\hat{\eta},t)=0,$$

and equations of evaluation (we disregard problems connected with operator ordering)

$$\dot{\hat{\eta}}=-\left.\{\eta,\epsilon\}_{D(\Phi)}\right|_{\eta=\hat{\eta}}=-\left.\{\eta,\Phi_l\}\,\{\Phi,\Phi\}^{-1}_{l,l'}\,\frac{\partial\Phi_{l'}}{\partial t}\right|_{\eta=\hat{\eta}}. \tag{3}$$

To each physical quantity A given in the Hamiltonian formalism by the function $A(\eta,t)$, we assign a Schrödinger operator $\hat{A}$ by the rule $\hat{A}=A(\hat{\eta},t)$; in the same manner we construct the quantum Hamiltonian $\hat{H}$, according to the classical Hamiltonian $H(\eta,t)$. The time evaluation of the state vector ψ in the Schrödinger picture is determined by the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=\hat{H}\,\psi,\ \ \hat{H}=H(\hat{\eta},t). \tag{4}$$

From (3) it follows, in particular, that

$$\frac{d\hat{A}}{dt}=\left.\{A,\epsilon\}_{D(\Phi)}\right|_{\eta=\hat{\eta}}$$

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and, as a consequence of (2), for arbitrary Schrödinger operators $\hat{A}$, $\hat{B}$ we have

$$[\hat{A}, \hat{B}\} = i\{A, B\}_{D(\Phi)}\big|_{\eta=\overset{\wedge}{\eta}} .$$

It is possible to see that quantum theories, which correspond to different initial data for the equation (3), are equivalent.

One can adduce some arguments in favor of the proposed quantization procedure. For instance, to check that the correspondence principle between classical and quantum equations of motion holds true in this procedure, we pass over to the Heisenberg representation, whose operators $\overset{\vee}{\eta}$ are related to the operators $\overset{\wedge}{\eta}$ as $\overset{\vee}{\eta}= U^{-1}\ \overset{\wedge}{\eta}\ U$, where U is the operator of the evolution of the Schrödinger equation,

$$i\frac{\partial U}{\partial t} =\hat{H} U, \quad U|_{t=0} = 1.$$

Heisenberg operator $\overset{\vee}{A}$ of an arbitrary physical quantity A is constructed from the corresponding Schrödinger operator $\hat{A}$ in the same manner $\overset{\vee}{A}= U^{-1}\ \hat{A}\ U$. One can find the total time derivative of the Heisenberg operator $\overset{\vee}{A}$:

$$\frac{d\overset{\vee}{A}}{dt} = \{A, H + \epsilon\}_{D(\Phi)}\big|_{\eta=\overset{\vee}{\eta}} , \tag{5}$$

which coincides in form with the classical equation of motion.

It follows from (5) that the Heisenberg operators $\overset{\vee}{\eta}$ also satisfy the equation

$$\dot{\overset{\vee}{\eta}}= \{\eta, H + \epsilon\}_{D(\Phi)}\big|_{\eta=\overset{\vee}{\eta}} . \tag{6}$$

Besides, one can easily verify that the equal-time relations hold for these operators,

$$[\overset{\vee}{\eta}, \overset{\vee}{\eta}' \} = i\{\eta, \eta'\}_{D(\Phi)}\big|_{\eta=\overset{\vee}{\eta}} , \quad \Phi(\overset{\vee}{\eta}, t) = 0.$$

These relations, together with (6), may be regarded as a prescription of the quantization in the Heisenberg picture for theories with time-dependent second-class constraints.

Note that the time dependence of Heisenberg operators in the theories considered is not unitary in the general case. In other words, no such (Hamiltonian) operator exists, whose commutator with a physical quantity would give its total time derivative. This is explained by the existence of two factors which determine the time evolution of the Heisenberg operator. The first one is the unitary evolution of the state vector in the Schrödinger picture, while the second one is the time variation of Schrödinger operators $\overset{\wedge}{\eta}$, which in the general case is non-unitary. Physically, this is explained by the fact that the dynamics develops on a surface which itself changes with time.

III. ALTERNATIVE APPROACH TO QUANTIZATION OF SYSTEMS WITH TIME-DEPENDENT CONSTRAINTS

Let us show that the equation (1) arises naturally from a consideration of a modified formulation, which is trivial non-stationary.

Let $L = L(q, \dot{q}, t)$ be a time-dependent Lagrangian of some singular theory, ($q = (q_1, ..., q_n)$, $\dot{q} = \frac{dq}{dt}$). One can consider another Lagrangian $L' = L'(q, \dot{q}, \dot{\tau}, \zeta, t)$, which depends on two supplementary variables τ, ζ and connected with origin Lagrangian L as:

$$L' =\tilde{L} +\zeta(\tau - t), \quad \tilde{L}= L(q, \dot{q}, \tau). \tag{7}$$

The theory with Lagrangian L' is equivalent to the theory with Lagrangian L in the sector of variables q. Indeed, the Lagrange equations in the theory with L' have the form:

$$\frac{\delta L'}{\delta q} = \frac{\partial \tilde{L}}{\partial q} - \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{q}} = 0, \tag{8}$$

$$\frac{\delta L'}{\delta \tau} = \zeta + \frac{\partial \tilde{L}}{\partial \tau} = 0, \quad \frac{\delta L'}{\delta \zeta} = \tau - t = 0. \tag{9}$$

Taking (9) into account in (8), it is easy to derive the equations:

$$\frac{\delta L}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0.$$

Let us consider the Hamiltonian formulation of the theory with Lagrangian L'. Introduce momenta

$$p = \frac{\partial L'}{\partial \dot{q}} = \frac{\partial \tilde{L}}{\partial \dot{q}}, \quad \epsilon = \frac{\partial L'}{\partial \dot{\tau}} = 0, \quad k = \frac{\partial L'}{\partial \dot{\zeta}} = 0. \tag{10}$$

From the relation (10) one can find primarily expressible velocities $\dot{X}$ and primary constraints $\Phi^{(1)}$, ($q = (X, x)$, $\dot{x}= \lambda$), $\dot{X}= V(q, p, \lambda, \tau)$, $\tilde{\Phi}^{(1)} = 0$, where $\tilde{\Phi}^{(1)} = \Phi^{(1)}(q, p, \tau)$. $\Phi^{(1)}(q, p, t)$ are constraints in the theory with Lagrangian L. Then Hamiltonian $H^{(1)'}$ has the form:

$$H^{(1)'}=(\frac{\partial L'}{\partial \dot{q}}\dot{q}-\widetilde{L}-\zeta(\tau-t))_{X=V}=\widetilde{H}-\zeta(\tau-t)$$

$$+\lambda\overset{\sim(1)}{\Phi}+\lambda_\epsilon\epsilon+\lambda_k k, \qquad (11)$$

where $\widetilde{H}=H(q,p,\tau)$. $H(q,p,t)$ is the Hamiltonian of a theory with Lagrangian L.

The condition of conservation of constraint $k=0$ in time gives

$$\dot{k}=\{k,H^{(1)\prime}\}=-\frac{\partial H^{(1)\prime}}{\partial\zeta}=\tau-t=0.$$

Thus, the secondary constraint $\Phi_1^{(2)}=\tau-t$ appears. Considering a condition of its conservation, we define λ_ϵ:

$$\dot{\Phi}_1^{(2)}=\frac{\partial\Phi_1^{(2)}}{\partial t}+\{\Phi_1^{(2)},H^{(1)\prime}\}=-1+\lambda_\epsilon=0,\quad \lambda_\epsilon=1.$$

From the condition of the constraint $\epsilon=0$ we get

$$\dot{\epsilon}=\{\epsilon,H^{(1)\prime}\}=-\frac{\partial H^{(1)\prime}}{\partial\tau}=-\frac{\partial\widetilde{H}}{\partial\tau}+\zeta-\lambda\frac{\partial\overset{\sim(1)}{\Phi}}{\partial\tau}=0,$$

or $\zeta=f(q,p,\tau,\lambda)$. From the condition of this constraint conservation in time we find $\lambda_k=\varphi(q,p,\lambda,\dot{\lambda})$, where f and φ are some functions.

Substituting the Lagrange multipliers found in the Hamiltonian (11), one transforms it to the form:

$$H^{(1)\prime}=\widetilde{H}-\zeta(\tau-t)+\lambda\overset{\sim(1)}{\Phi}+\varphi k+\epsilon. \qquad (12)$$

Further, one can continue the Dirac procedure to obtain λ- multipliers and secondary constraints already with Hamiltonian (12) [3]. Using the Dirac procedure for the constraints $\overset{\sim(1)}{\Phi}$, one can use the following Hamiltonian

$$H^{(1)\prime}=\widetilde{H}_{eff}+\lambda\overset{\sim(1)}{\Phi},$$

instead of the Hamiltonian (12), where $\widetilde{H}_{eff}=\widetilde{H}+\epsilon$. This is so because of the constraints $\overset{\sim(1)}{\Phi}$ and secondary constraints, which can appear, do not contain the variables ζ and k. Let $\overset{\sim(2)}{\Phi}$ be secondary constraints. Then,

$$\overset{\sim(2)}{\Phi}=\Phi^{(2)}(q,p,\tau),$$

where $\Phi^{(2)}(q,p,t)$ are secondary constraints of the theory with Lagrangian L.

Suppose that $\Phi=(\Phi^{(1)},\Phi^{(2)})$ is a total system of constraints of the theory with Lagrangian L and is of second class. Then the total system of constraints of the theory with Lagrangian L' is also of second class and has the form

$$\widetilde{\Phi}=0,\quad \tau-t=0,\quad \epsilon=0,$$

$$\zeta-f=0,\quad k=0,\quad \widetilde{\Phi}=(\overset{\sim(1)}{\Phi},\overset{\sim(2)}{\Phi}).$$

Because the constraints for the variables ζ and k have the special forms [3], they can be merely excluded from the equations of motion by means of the constraints. In doing this, one can easily discover that the equations for the rest variables coincide with equation (1).

The quantization in Schrödinger picture of the theory in question may be formulated in the following form. We have a Hamiltonian theory with canonical variables $\eta=(q,p)$ and (t,ϵ). The surface of constraints is described by the equation

$$\Phi(\eta,t)=0.$$

The Hamiltonian of the theory is H. When quantizing, all variables became operators with commutation relations

$$[\hat{Q},\hat{Q}']=i\{Q,Q'\}_{D(\Phi)}\big|_{Q=\hat{Q}},\quad Q=(\eta;t,\epsilon). \qquad (13)$$

Constraints are equal to zero,

$$\Phi(\hat{\eta},t)=0,$$

and conditions on the state-vectors hold (similar to the Dirac quantization of theories with first class constraints [2]),

$$H_{eff}\psi=0\ ,\quad H_{eff}=H+\epsilon. \qquad (14)$$

One can verify that this quantization is fully equivalent to the GT-quantization in the Schrödinger picture. Indeed, let us realize $\hat{t}$ as operator of multiplication by the variable t. Then,

$$\hat{\epsilon}=-i\frac{\partial}{\partial t}.$$

From (11) we have

$$[\hat{\eta},\hat{t}]_-=0\,;\quad [\hat{\eta},\hat{\epsilon}]_-=i\{\eta,\epsilon\}_{D(\Phi)}\big|_{\eta=\hat{\eta}}. \qquad (15)$$

And in the selected realization,

$$[\hat{\eta},\hat{\epsilon}]_-=i\,\hat{\tau}, \qquad (16)$$

the condition (3) follows from (15) and (16). Finally, the condition (14) is the Schrödinger equation.

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